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A Numerical Approximation Theory for Second-Order Integral Differential Equations

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This paper is a companion paper to "An Oscillation Theory for Second-Order Integral Differential Equations." The underlying theme is that both topic (oscillation theory and numerical oscillation theory) follow as a corollary to an approximation theory of quadratic forms given previously by the author. In Section I we give the mathematical preliminaries; some of which are not included in the earlier paper. This includes the relationship between the fundamental quadratic form and its integral differential equation (the Euler–Lagrange equations). In Section II the approximating quadratic forms are defined on the approximating Hilbert space. In Section III we show that our approximating hypothesis are satisfied and give the fundamental inequality relationships [Eqs. (12) and (13)]. We also show that the mth oscillation point is a continuous function of our approximating parameter. Finally in Section IV we show how that the approximating indices may be easily obtained by computer algorithms.

I. PRELIMINARIES

We note that a major part of our preliminaries are given in Section I of [4]. To avoid needless (and expensive) duplication we will start with Theorem 4 and Eq. (8). Thus Theorems 1, 2, and 3 and Eqs. (1)–(8) are as given in [4].

The fundamental quadratic form is $J(x) = J(x, x)$, where

$$J(x, y) = \int_a^b R_{ij}(t)\, x^i(t)\, y^j(t)\, dt + \int_a^b \int_a^b K_{ij}(s, t)\, x^i(s)\, y^j(t)\, ds\, dt. \tag{8}$$

The fundamental differential equation is the Euler–Lagrange equation for (8), namely

$$\tau_1'(t) = \tau_0(t), \tag{9}$$

where

$$\tau_0(t) = R_{00}(t)\, x(t) + R_{10}(t)\, x'(t) + \int_a^b [K_{00}(s, t)\, x(s) + K_{10}(s, t)\, x'(s)]\, ds \tag{10a}$$

and

$$\tau_1(t) = R_{01}(t)\, x(t) + R_{11}(t)\, x'(t) + \int_a^b [K_{01}(s, t)\, x(s) + K_{11}(s, t)\, x'(s)]\, ds. \tag{10b}$$

The conditions on (8), (9), and (10) are given in [4]. We note that $i, j = 0, 1$; $R_{01}(t) = R_{10}(t)$, $K_{ij}(s, t) = K_{ji}(t, s)$; $x^0(t) = x(t)$ and $x^1(t) = x'(t)$ (the derivative of $x(t)$).

II. The Forms $J(x; \mu)$ and the Spaces $\mathscr{B}(\mu)$

In this section the approximating quadratic form $J(x; \mu)$ on the space $\mathscr{B}(\mu)$ is defined.

Let $\mathscr{A}$ denote the arcs $x(t)$ which are absolutely continuous on $\Lambda = [a, b]$ and have square integrable derivates $x'(t)$. The norm, $\| x \| = (x, x)^{1/2}$ on the Hilbert space $\mathscr{A}$ is defined from

$$(x, y) = x(a)\, y(a) + \int_a^b x'(t)\, y'(t)\, dt.$$

Let Σ denote *the set of real numbers of the form* $\sigma = 1/n$ $(n = 1, 2,...)$ *and zero*. The metric on Σ is the absolute value function. Let $\sigma = 1/n$, define the partition

$$\pi(\sigma) = (a_0 = a < a_1 < a_2 \cdots < a_n = b),$$

where

$$a_k = k\,\frac{b-a}{n} + a \qquad (k = 0,..., n). \tag{11}$$

The space $\mathscr{A}(\sigma)$ *is the set of continuous broken linear functions with vertices at* $\pi(\sigma)$. Let $\mathscr{A}(0)$ denote the subset of $\mathscr{A}$ satisfying $x(a) = 0$ and $x(b) = 0$.

For each λ in Λ let $\mathscr{H}(\lambda)$ denote the arcs $x(t)$ in $\mathscr{A}$ satisfying $x(a) = 0$ and $x(t) \equiv 0$ on $[\lambda, b]$. We note that $\mathscr{H}(\lambda)$ has the properties of [4]. Finally if $\mu = (\lambda, \sigma)$ is in the metric space $\mathscr{M} = \Lambda \times \Sigma$, let $\mathscr{B}(\mu) = \mathscr{A}(\sigma) \cap \mathscr{H}(\lambda)$. Thus an arc $x(t)$ in $\mathscr{B}(\lambda, \sigma)$ *is a spline of degree* 2 *on* $[a, a_k]$ *where* $a_k \leqslant \lambda < a_{k+1}$, *such that* $x(a) = 0$ *and* $x(t) \equiv 0$ *on* $[a_k, b]$.

To construct $J(x; \mu)$: Let $R_{ij\sigma}(t) = R_{ij}(a_k)$ and $K_{ij\sigma}(s, t) = K_{ij}(a_e, a_k)$ if t in $[a_k, a_{k+1})$ and s in $[a_e, a_{e+1})$. Let $R_{ij\sigma}(b) = R_{ij}(b)$ and $K_{ij\sigma}(b, t) = K_{ij}(b, b)$. Finally for $\mu = (\lambda, \sigma)$ let $J(x; \mu) = J(x, x; \mu)$, where

$$J(x, y; \mu) = \int_a^b R_{ij\sigma}(t)\, x^i(t)\, y^j(t)\, dt + \int_a^b \int_a^b K_{ij\sigma}(s, t)\, x^i(s)\, y^j(t)\, ds\, dt \tag{12}$$

is defined for arcs $x(t)$, $y(t)$ in $\mathscr{B}(\mu)$.

III. Inequalities

In this section we show that Theorems 1, 2, and 3 hold in our setting. The main results are contained in Theorem 6. Theorem 7 shows that the mth oscillation point $\lambda_m(\sigma)$ is a continuous function of σ. We begin by showing that the hypothesis for Theorems 1, 2, and 3 hold in our setting.

THEOREM 4. *If $\mathscr{A}(\sigma)$ is the set of continuous broken linear functions defined in Section* 2, *then $\mathscr{A}(\sigma)$ satisfies hypothesis* (1a) *and* (1b) *in* [4].

The proof is a straightforward application of the properties of 2-splines. A proof by more conventional variational methods may also be found in [3].

THEOREM 5. *For each σ in Σ let $J(x; \sigma)$ and $\mathscr{A}(\sigma)$ be defined as in Section* 2, *then conditions* (2a), (2b), *and* (2c) *hold.*

The theorem with $K_{ij}(s, t) \equiv 0$ has been given in [3]. From [6, p. 49] or straightforward calculations we note that

$$\int_a^b \int_a^b K_{ij}(s, t)\, x^i(s)\, x^j(s)\, ds\, dt$$

is a compact (completely continuous) quadratic form on the Hilbert space $\mathscr{A}$. The theorem now follows by a minor modification in [3, Theorem 10].

For the next theorem let $s(\lambda, 0)$ and $n(\lambda, 0)$ be the index and nullity of $J(x; 0) = J(x)$ given in (8) defined on the space $\mathscr{H}(\lambda)$. For $\mu = (\lambda, \sigma)$, $\sigma \neq 0$ let $s(\mu)$ and $n(\mu)$ be the index and nullity of $J(x; \mu)$ defined on $\mathscr{A}(\mu)$ given in (12). Let d be the metric on $\mathscr{M}$ given by

$$d(\mu_1, \mu_2) = | \lambda_2 - \lambda_1 | + | \sigma_2 - \sigma_1 |$$

where for $i = 1, 2$, $\mu_i = (\lambda_i, \sigma_i)$ in $\mathscr{M} = \Lambda \times \Sigma$.

THEOREM 6. *For any $\mu_0 = (\lambda_0, \sigma_0)$ in $\mathscr{M}$ there exists $\delta > 0$ such that if $\mu = (\lambda, \sigma)$ in $\mathscr{M}$, $d(\mu_0, \mu) < \delta$ then*

$$s(\lambda_0, \sigma_0) \leqslant s(\lambda, \sigma) \leqslant s(\lambda, \sigma) + n(\lambda, \sigma) \leqslant s(\lambda_0, \sigma_0) + n(\lambda_0, \sigma_0). \tag{13}$$

Furthermore

$$n(\lambda_0, \sigma_0) = 0 \quad \textit{implies} \quad s(\lambda, \sigma) = s(\lambda_0, \sigma_0) \quad \textit{and} \quad n(\lambda, \sigma) = 0. \tag{14}$$

The proof follows directly since Theorems 4 and 5 are the hypothesis for Theorem 3 in this setting.

Let σ in Σ be given. A point λ at which $s(\lambda, \sigma)$ is discontinuous will be called an *oscillation point* of $J(x; \sigma)$ relative to $\{\mathscr{H}(\lambda): \lambda \text{ in } \Lambda\}$. The oscillation points will be denoted by $\lambda_m(\sigma)$. Thus $a < \lambda_1 \leqslant \lambda_2 < \cdots < b$. The concept of oscillation in our setting, and the fact that our definition generalizes the usual definition of oscillation, have been given in [4].

For the next theorem we assume the quadratic form in (8) is normal. That is $n(\lambda, 0) \neq 0$ only at a finite number of points, or equivalently the only solution to (9) satisfying $x(t_0) = x'(t_0) = 0$ for $a \leqslant t_0 \leqslant b$ is the zero solution.

THEOREM 7. *The* mth *oscillation point* $\lambda_m(\sigma)$ *is a continuous function of* σ $(m = 1, 2, 3, \ldots)$ *if* $\lambda_m(\sigma) < b$.

The theorem is trivial if $\sigma \neq 0$ as in this case $\sigma = 1/n$ for some integer n. If $\sigma = 0$ let $\lambda_1(0)$ denote the first oscillation point. Choose $\epsilon > 0$. Then

$$n(\lambda_1(0) - \epsilon, 0) = 0, \qquad s(\lambda_1(0) - \epsilon, 0) = 0,$$
$$n(\lambda_1(0) + \epsilon, 0) = 0, \qquad s(\lambda_1(0) + \epsilon, 0) = 1,$$

and hence by (14) there exists $\delta > 0$ such that if $|\sigma| < \delta$ we have

$$n(\lambda_1(0) - \epsilon, \sigma) = 0, \qquad s(\lambda_1(0) - \epsilon, \sigma) = 0,$$
$$n(\lambda_1(0) + \epsilon, \sigma) = 0, \qquad \text{and} \qquad s(\lambda_1(0) + \epsilon, \sigma) = 1.$$

That is $|\lambda_1(\sigma) - \lambda_1(0)| < 2\epsilon$ whenever $|\sigma| < \delta$. Since $s(b, 0) = M < \infty$ δ may be chosen independent of m and we are done.

IV. THE MATRIX $d_{\alpha\beta}(\mu)$

In this section we show that the indices $s(\lambda, \sigma)$ and $n(\lambda, \sigma)$ for $\sigma \neq 0$ are the number of negative and zero eigenvalues of a real symmetric tridiagonal matrix.

For $\sigma = 1/n$ the space $\mathscr{A}(\sigma)$ is an $n - 1$ whose basis may be choosen as $z_1(t), z_2(t), \ldots, z_{n-1}(t)$, where

$$z_k(t) = \begin{cases} 1 - \{n(t - a_k)/(b - a)\} & \text{if } \ t \text{ in } [a_{k+1}, a_{k-1}], \\ 0 & \text{otherwise.} \end{cases}$$

Let $x(t) = a_\alpha z_\alpha(t)$ be in $\mathscr{B}(\mu)$ for $\mu = (\lambda, \sigma)$, $\sigma \neq 0$. Then

$$\begin{aligned} J(x; \mu) &= \int_a^b R_{ij\sigma}(t)\, x^i(t)\, x^j(t)\, dt + \int_a^b \int_a^b K_{ij\sigma}(s, t)\, x^i(s)\, x^j(t)\, ds\, dt \\ &= a_\alpha a_\beta \left\{ \int_a^b R_{ij\sigma}(t)\, z_\alpha{}^i(t)\, z_\beta{}^j(t)\, dt + \int_a^b \int_a^b K_{ij\sigma}(s, t)\, z_\alpha{}^i(s)\, z_\beta{}^j(t)\, ds\, dt \right\} \\ &= a_\alpha a_\beta J(z_\alpha, z_\beta; \mu) = a_\alpha a_\beta D_{\alpha\beta}(\mu), \end{aligned}$$

where $\alpha, \beta = 1,..., n$. Thus $s(\mu)$ and $n(\mu)$ are the number of negative and zero eigenvalues of the real, symmetric matrix $D_{\alpha\beta}(\mu)$. Furthermore $D_{\alpha\beta}(\mu)$ is tridiagonal since

$$D_{\alpha\beta}(\mu) = J(z_\alpha, z_\beta; \mu) = 0 \qquad \text{if} \qquad |\alpha - \beta| \geqslant 2.$$

We note that our methods are very practical for computer methods: If $\sigma \neq 0$ the calculation of the elements $D_{\alpha\beta}(\mu) = J(z_\alpha, z_\beta; \mu)$ can be found in milliseconds due to our choice of $z_\alpha(t)$, $R_{ij\sigma}(t)$, etc. Furthermore if $\mu = (\lambda, \sigma)$, $a_k \leqslant \lambda < a_{k+1}$ $(k = 1,..., n)$, then $D_{\alpha\beta}(\mu)$ is a $k \times k$ matrix, "increasing" in λ so that the upper $k \times k$ submatrix of $D_{\alpha\beta}(a_{k+1}, \sigma)$ is $D_{\alpha\beta}(a_k, \sigma)$. The index $m(\lambda, \sigma) = s(\lambda, \sigma) + n(\lambda, \sigma)$ may be found by elementary Sturm sequence type arguments as it represents the number of eigenvalues λ satisfying $-\infty < \lambda \leqslant 0$ given by the quantity $V(0) - V(\infty)$ in Theorem 8.3 of [7]. This calculation is left as an exercise for the interested reader; details may be found in [7, pp. 494–495].

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